

PYTHAGOREAN TRIPLETS-ASSOCIATED DUAL TRIPLETS AND ALGEBRAIC PROPERTIES, FERMAT POINT, NAPOLEON POINT AND EXTENSION OF FERMAT PROPERTY FOR PYTHAGOREAN TRIPLETS

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ABSTRACT

This paper draws some important points closely related to Pythagorean triplets by defining dual of a given triplet and then by establishing some algebraic properties between two or more triplets. It further details calculating Fermat point and Napoleon point in right triangles of Fermat family-- with sides observing Fermat properties (sides of a right triangle in the form $(a, b = a + 1, h)$). General formulae for Fermat and Napoleon points have been derived they align with mathematical properties inherent in these said points. Finally, we have established a systematic approach of extending Fermat properties to some consecutive positive integers and identified mathematical symmetry by introducing its general format.

KEYWORDS: Fermat Triangle, Fermat Triplets, Fermat Point, Napoleon Point, Dual of A Triplet, Extension of Fermat Triplets & NSW Sequence.

Abbreviation and Notations

Abbreviation: E-Pythagorean Property, NSW (Newman Shank William) Sequence.

Notation: $SD(T_i) = T_i^{th}$ set of duals of a given triplet

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INTRODUCTION

PRIMITIVE TRIPLET AND SET OF ITS DUALS

We consider a primitive* form (a, b, h) of a Pythagorean triplet.

[A Pythagorean triplet $T = (a, b, h)$, a, b , and $h \in N$, is said to be a primitive one if it satisfies the following properties. (1) $(a, b) = 1$ (2) $a < b < h$ (3) $a^2 + b^2 = h^2$]

We introduce a break-up of sides **a** and **b** of a right triangle as

Let a triplet $T_1 = (a, b, h)$ and $a = m_1 \times m_2$, $b = n_1 \times n_2$ (refer (2) * below) then in correspondence to Pythagorean Property $a^2 + b^2 = h^2$ we can write $(m_1 \times m_2)^2 + (n_1 \times n_2)^2 = h^2$; this in turn implies that we have four different sets of triplets originating from this relation. They are as follows.

$$\begin{aligned} (1) T_{11} &= (m_1 / n_1, n_2 / m_2, h / (n_1 \times m_2)) & (2) T_{12} &= (m_1 / n_2, n_1 / m_2, h / (n_2 \times m_2)) \\ (3) T_{13} &= (m_2 / n_1, n_2 / m_1, h / (n_1 \times m_1)) & (4) T_{14} &= (m_2 / n_2, n_1 / m_1, h / (n_2 \times m_1)) \end{aligned} \quad \text{set (1)}$$

Each bracketed triplet above is referred as the **dual** of the original triplet (a, b, h) as it satisfies a property

parallel to Pythagorean property for integers.

$$\text{E.G. } (m_1/n_1)^2 + (n_2/m_2)^2 = (h/(n_1 \times m_2))^2 \text{ etc....}$$

We shall now and in the remaining part of the paper refer to this property as an **E- Pythagorean** property.

We, at this stage, note two important points;

- (1) That expressing integers **a** and **b** as $a = m_2 \times m_1$ and $b = n_2 \times n_1$ will not change the number of triplets and the format of triplet except the order of triplets appearing in the original form as written above.
- (2) To a set of integers **a**, and **b** there can be finitely many break-ups and we shall record them by the corresponding notation T_i for $i = 1, 2, 3, \dots$

We denote the set of all triplets generated by the format T_1 as **SD (T_1)**

i.e. **SD (T_1)** = { $T_{11}, T_{12}, T_{13}, T_{14}$ }, where T_{11}, T_{12}, T_{13} , and T_{14} are shown above in set (1).

In the same way we may write the members of the same triplet (a, b, h) in some different formats, calling them as T_2, T_3, \dots etc.

Say **$T_2: (a, b, h)$** where $a = p_1 \times p_2, b = q_1 \times q_2$, and **h**

Corresponding to this triplet format also we have four other triplets as discussed above.

We denote these as T_{21}, T_{22}, T_{23} , and T_{24} . The set of all such triplets according to the format T_2 is denoted by **SD (T_2)**.

$$(1) T_{21} = (p_1/q_1, q_2/p_2, h/(q_1 \times p_2)) \quad (2) T_{22} = (p_1/q_2, q_1/p_2, h/(q_2 \times p_2))$$

$$(3) T_{23} = (p_2/q_1, q_2/p_1, h/(q_1 \times p_1)) \quad (4) T_{24} = (p_2/q_2, q_1/p_1, h/(q_2 \times p_1))$$

$$\text{i.e. } \mathbf{SD (T_2)} = \{T_{21}, T_{22}, T_{23}, T_{24}\} \quad (2)$$

We note the fact that each set **SD (T_i)** has exactly four members of dual triplets. If the integer **a** can be expressed in m different non-commutative ways and to each way of that the integer **b** has n different non-commutative ways of expression, then the total number sets of dual triplets will be $m \times n$; and to each one of $m \times n$ there shall correspond four triplets as shown above. This implies that there shall be $m \times n$ sets of the type **SD (T_i)** resulting into $4mn$ different triplets including the given one. [In case if either **a** or **b** is a perfect square then there will be $(4mn - 3)$ number of different dual triplets.]

We illustrate this by an example.

Let us consider a triplet (5, 12, 13) and associated different sets of dual triplets which be shown in different dual formats.

$$(1) T_1: a = 5 = 1 \times 5 \quad b = 12 = 1 \times 12 \quad \text{and } h = 13$$

Corresponding to this break-up

$$\mathbf{SD (T_1)} = \{T_{11} = (1/1, 12/5, 13/5), T_{12} = (1/12, 1/5, 13/60), T_{13} = (5/1, 12/1, 13/1), T_{14} = (5/12, 1/1, 13/12)\}$$

[The triplet T_{13} is same as that of the given triplet (5, 12, 13)]

(2) T_2 : $a = 5 = 1 \times 5$ $b = 2 \times 6$ and $h = 13$

$$SD(T_2) = \{T_{21} = (1/2, 6/5, 13/10), T_{22} = (1/6, 2/5, 13/30), T_{23} = (5/2, 6/1, 13/2), T_{24} = (5/6, 2/1, 13/6)\}$$

(3) T_3 : $a = 5 = 1 \times 5$ $b = 3 \times 4$ and $h = 13$

$$SD(T_3) = \{T_{31} = (1/3, 4/5, 13/15), T_{32} = (1/4, 3/5, 13/20), T_{33} = (5/3, 4/1, 13/3), T_{34} = (5/4, 3/1, 13/4)\}$$

As discussed above, $a = 5$ has only one way [$5 = 1 \times 5$] of expression and to each way of that $b = 12$ has 3 non-commutative ways of expression then there are $1 \times 3 = 3$ sets like $SD(T_i)$ for $i = 1, 2, 3$ and to each one of these set there are 4 associated triplets. This results into 12 different triplets.

ALGEBRAIC OPERATIONS ON SETS OF DUALS

At this stage we introduce algebraic operations (1) addition and (2) subtraction on the different triplets T_{ij} . This has been further sub-divided in to two parts. In the first part we perform operations on the members of the same set $SD(T_i)$ and in the second part we inter- relate triplets of the different sets like $SD(T_i)$ and carry out algebraic operations on its members.

Addition and Subtraction on the Triplets of the Same Set

In this section we introduce algebraic operation addition and subtraction on the members of the same set and assert that the resultant also preserves E-Pythagorean nature.

Let us consider any two member triplets of any set; say $SD(T_1)$.

Consider T_{11} and T_{12} .

$$T_{11} = (m_1/n_1, n_2/m_2, h/(n_1 \times m_2)) \text{ and } T_{12} = (m_1/n_2, n_1/m_2, h/(n_2 \times m_2))$$

From the basic property we have the two relations as follows.

$$\left(\frac{m_1}{n_1}\right)^2 + \left(\frac{n_2}{m_2}\right)^2 = \left(\frac{h}{n_1 \times m_2}\right)^2 \text{ and } \left(\frac{m_1}{n_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 = \left(\frac{h}{n_2 \times m_2}\right)^2$$

What we now claim that

(1) $T_{11} + T_{12}$ with addition of corresponding elements as usual gives a triplet and its components also satisfy E-Pythagorean property.

i.e. $T_{11} + T_{12} = (m_1/n_1 + m_1/n_2, n_2/m_2 + n_1/m_2, h/(n_1 \times m_2) + h/(n_2 \times m_2))$ which, as claimed, will satisfy

$$\left(\frac{m_1}{n_1} + \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} + \frac{n_1}{m_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} + \frac{1}{(n_2 \times m_2)}\right)^2 \quad (3)$$

[This result is proved in the **annexure-1**]

In addition to this, we prove that the subtraction of corresponding elements of the two triplets gives rise to a new triplet which also satisfies E-Pythagorean property.

(2) To show that $T_{11} - T_{12}$ is also an E-Pythagorean triplet.

$$T_{11} - T_{12} = (|m_1/n_1 - m_1/n_2|, |n_2/m_2 - n_1/m_2|, |h/(n_1 \times m_2) - h/(n_2 \times m_2)|) \text{ which satisfies E-Pythagorean property.}$$

$$\text{i.e. } \left(\frac{m_1}{n_1} - \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} - \frac{n_1}{m_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} - \frac{1}{(n_2 \times m_2)}\right)^2 \quad (4)$$

[This result is proved in the **annexure-2**]

Example

We take an illustration from the example cited in the previous section.

$$\text{Let } T_{11} = (1/1, 12/5, 13/5), T_{12} = (1/12, 1/5, 13/60)$$

$$T_{11} + T_{12} = (1/1 + 1/12, 12/5 + 1/5, 13/5 + 13/60)$$

= (13/12, 13/5, 13x13/60), which is E-Pythagorean Triplets.

$$\& T_{11} - T_{12} = (|1/1 - 1/12|, |12/5 - 1/5|, |13/5 - 13/60|)$$

= (11/12, 11/5, 13x11/60), which is E-Pythagorean Triplets.

Addition and Subtraction on the Triplets of Different Sets

In this section we introduce algebraic operation addition and subtraction on the members of the different sets, say **SD (T₁)** and **SD (T₂)**. Continuing in the same design and keeping corresponding component-wise addition and subtraction; we get two different triplets and here too we preserve our claim that elements of the resultant triplets also preserves E-Pythagorean property.

Let us consider any two member triplets of any set; say **SD (T₁)** and **SD (T₂)**

Consider T₁₁ and T₂₁; the first one is the members of **SD (T₁)** and the second one is from **SD (T₂)**.

$$T_{11} = (m_1/n_1, n_2/m_2, h/(n_1 \times m_2)) \text{ and } T_{21} = (p_1/q_1, q_2/p_2, h/(q_1 \times p_2))$$

We know that both are derived from the basic relation $a^2 + b^2 = h^2$.

$$(m_1 \times m_2)^2 + (n_1 \times n_2)^2 = h^2 \text{ and } (p_1 \times p_2)^2 + (q_1 \times q_2)^2 = h^2$$

We now write $T_{11} + T_{21} = (m_1/n_1 + p_1/q_1, n_2/m_2 + q_2/p_2, h/(n_1 \times m_2) + h/(q_1 \times p_2))$ and

$$T_{11} - T_{21} = (|m_1/n_1 - p_1/q_1|, |n_2/m_2 - q_2/p_2|, |h/(n_1 \times m_2) - h/(q_1 \times p_2)|)$$

We claim that the results of $T_{11} + T_{21}$ and $T_{11} - T_{21}$ are also triplets with their components satisfying

E-Pythagorean property.

We will prove that

$$(1) \left(\frac{m_1}{n_1} + \frac{p_1}{q_1}\right)^2 + \left(\frac{n_2}{m_2} + \frac{q_2}{p_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} + \frac{1}{(q_1 \times p_2)}\right)^2 \quad (5)$$

and

$$(2) \left(\frac{m_1}{n_1} - \frac{p_1}{q_1}\right)^2 + \left(\frac{n_2}{m_2} - \frac{q_2}{p_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} - \frac{1}{(q_1 \times p_2)}\right)^2 \quad (6)$$

[This result is proved in the annexure-3 and 4]

Example

$$\begin{aligned} \text{Let } T_{11} &= (1/1, 12/5, 13/5) \text{ \& } T_{21} = (1/2, 6/5, 13/10) \\ T_{11} + T_{21} &= (1/1 + 1/2, 12/5 + 6/5, 13/5 + 13/10) \\ &= (3/2, 18/5, 13 \times 3/10), \text{ which is E-Pythagorean triplets.} \\ \& T_{11} - T_{21} &= (|1/1 - 1/2|, |12/5 - 6/5|, |13/5 - 13/10|) \\ &= (1/2, 6/5, 13/10), \text{ which is E-Pythagorean Triplet.} \end{aligned}$$

BEYOND THE FERMAT TRIPLETS (FROM TRIPLET TO N TUPLE)

Introduction

It is known the Pythagorean relationship arrange the side a, b and h of a right triangle

$(a^2 + b^2 = h^2)$ is known as Fermat triplets if $b = a+1$.

e.g, $a = 20$, $b = a + 1 = 21$ and $h = 29$

There are infinitely many triplets possessing the above mentioned Fermat property. We have also identified algebraic relationship within the corresponding terms of each triplet. There are many trigonometric relations connecting the sides of a triplet and different geometrical features of corresponding members of right triangles of Fermat family.

In this system, we want to go beyond the system of triplets and extend the pattern.

In R^2 we have a right triangle with sides n and $n+1$ so that $n^2 + (n+1)^2 = h^2$ where $h > n+1$ for some $n \in \mathbb{N}$

This system of 3 integers (2 on left side and 1 on right side) is such that it is satisfied for

$$n_1 = 3 \text{ and } h_1 = 5$$

$$n_2 = 20 \text{ and } h_2 = 29$$

$$n_3 = 119 \text{ and } h_3 = 169 \dots \text{etc.}$$

We have the sum of terms n and $n + 1$ as found in the corresponding terms Newman-Shank-William sequence. Also each successive terms of hypotenuse h strictly follow a recurrence relation

$$h_n = 6 h_{n-1} - h_{n-2}$$

For all h_i of hypotenuses of right triangles of Fermat family.

Extending the Relation

The above mentioned equation has a special form when $h = n + 2$.

i.e., we have $n^2 + (n + 1)^2 = (n + 2)^2$ with 3 terms (2 on left side and 1 on right side). We wish to continue by adding such consecution of one term on each side.

$$\text{i.e., } n^2 + (n + 1)^2 + (n + 2)^2 = (n + 3)^2 + (n + 4)^2, \text{ true for } n = 10$$

in each succession we extend the set-up pursuing the same pattern

$$\text{i.e., } n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 = (n + 4)^2 + (n + 5)^2 + (n + 6)^2, \text{ true for } n = 21$$

We compress the data in a tabular form,

Table 1

Set-up	No. of Terms	Break-up		First Term of The Set Solution (N)
		Left	Right	
1	3	2	1	3
2	5	3	2	10
3	7	4	3	21
4	9	5	4	36
5	11	6	5	55
-	-	-	-	-
-	-	-	-	-
K	2k+1	k+1	K	k(2k+1)

Numerical format

Set-1 $3^2 + 4^2 = 5^2$ [3 Successive term beginning with 3]

Set-2 $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [5 Successive term beginning with 10]

Set-4 $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$ [7 Successive term beginning with 21]

so, on....

FERMAT POINT IN A RIGHT TRIANGLE OF FERMAT FAMILY

A Fermat point is a point within a triangle such that sum of its distances from three vertices is minimum.

In general, for a triangle ABC a point F is a Fermat point if $AF + BF + CF$ is minimum.

The Fermat point of a right triangle of Fermat family has certain characteristics which are discussed below.

- Construct an equilateral triangle, heading away from origin, on each sides of the given right triangle of Fermat family.
- Draw a line joining each new vertex of the equilateral triangle to the opposite vertex of the original Right triangle.
- All such lines are concurrent at a point called Fermat point

Let ABC be any given Fermat triangle. On the sides BC , CA , AB of the triangle, construct equilateral triangles as discussed above; let these be BCA' , ACB' , and ABC' respectively.

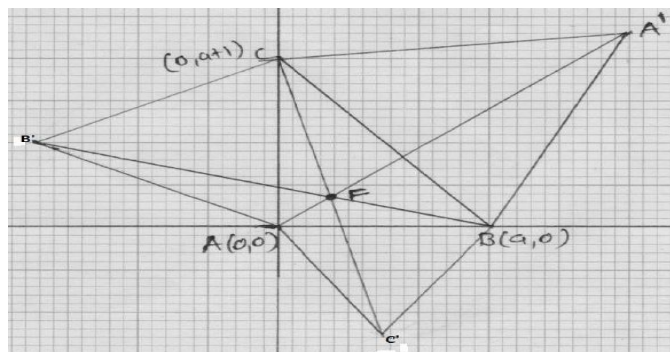


Figure 1: Fermat Point F of Right Angle Triangle ABC of Fermat Family

We have derived Fermat point for a Fermat right triangle with $A(0, 0)$, $B(a, 0)$, and $C(0, a+1)$

Fermat point of Fermat family is, $\left(\frac{ab(\sqrt{3}b+a)}{(6ab+2\sqrt{3}h^2)}, \frac{ab(\sqrt{3}a+b)}{(6ab+2\sqrt{3}h^2)}\right)$, where $b = a + 1$

$$\text{and } a^2 + b^2 = h^2 \quad (7)$$

The F, as said earlier is such that $AF + BF + CF$ is minimum. (8)

In addition to that the Points A' , B' , and C' in accordance with the given triangle ABC are such that

$$AA' = BB' = CC' = \sqrt{h^2 + \sqrt{3}ab}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2 \quad (9)$$

[The proof is shown derived in Annexure -5]

NAPOLEON POINT OF RIGHT TRIANGLE OF FERMAT FAMILY

Let ABC be any given Fermat right angle triangle. On the sides BC , CA , AB of the triangle, construct equilateral triangles $A'BC$, $AB'C$ and ABC' , in the direction not towards the origin $A(0, 0)$, respectively. Let the centroids of these corresponding triangles be P , Q and R . The line segments AP , BQ and CR are concurrent at a point N . The point of concurrence, N , is the Napoleon point of the triangle ABC . The triangle PQR is called the Napoleon triangle of the triangle ABC . Napoleon's theorem asserts that this triangle

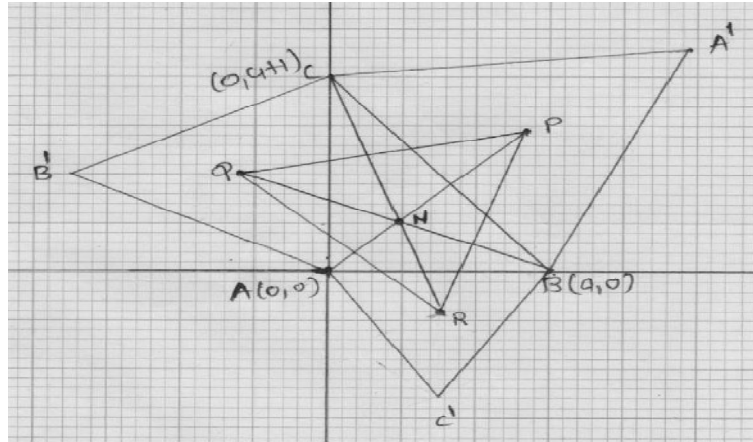


Figure 2: Napoleon Point N of Right Angle Triangle of Fermat Family

PQR is an equilateral triangle.

From section (1.4), we have $A = A(0,0)$, $B = B(a, 0)$, $C = C(0, b = a + 1)$,

$$A' = A'\left(\frac{(a(\sqrt{3}+1)+\sqrt{3})}{2}, \frac{(a(\sqrt{3}+1)+1)}{2}\right), \quad C' = C'\left(\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right), \quad B' = B'\left(-\frac{\sqrt{3}}{2}(a+1), \frac{1}{2}(a+1)\right).$$

Now we derive centroid P , Q & R of equilateral triangles BCA' , CAB' and ABC' respectively.

Napoleon point of Fermat family $N = N\left(\frac{\sqrt{3}ab(b+\sqrt{3}a)}{(10ab+2\sqrt{3}h^2)}, \frac{\sqrt{3}ab(a+\sqrt{3}b)}{(10ab+2\sqrt{3}h^2)}\right)$, Where $b = a + 1$

$$\text{and } a^2 + b^2 = h^2. \quad (10)$$

We add that

$$PQ = QR = PR = \sqrt{\frac{h^2 + \sqrt{3}ab}{3}}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2.$$

[The proof is shown derived in **annexure -6**]

CENTROID OF NAPOLEON TRIANGLE

Geometrical properties continue to grow and as a result of careful diagnosis, we have searched for finding centroid of the Napoleon equilateral triangle PQR. We have, on hand the coordinates of the points P, Q, and R which helps us derive the centroid $G = (a/3, b/3)$ where $b = a + 1$.

This leaves a set of three points F – the Fermat Point, N – the Napoleon Point and G - Centroid of the Napoleon triangle.

An interesting point that we have observed is that the centroid of the original Fermat triangle is the centroid of its associated Napoleon triangle.

A comprehensive figure clearly indicating existence of all these is shown below in figure 3.

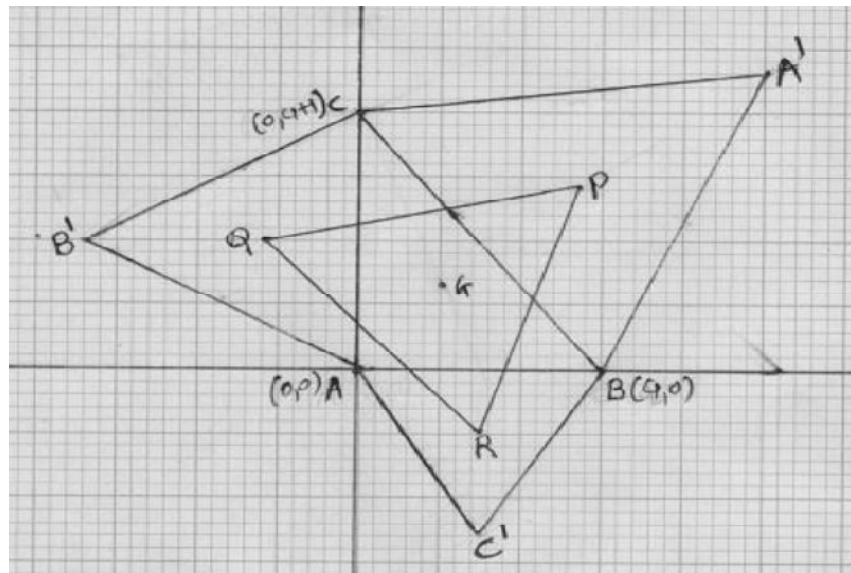


Figure 3: G-Centroid of Napoleon Triangle of Right Angle Triangle of Fermat Family

CONCLUSIONS

Resting on the same base of centuries old but lively Pythagorean and Fermat work and related notions, this paper opens wide many new avenues extending the geometrical dimensions. We are of the opinion that the set of Duals to a given triplet and its algebraic properties, as we have observed and derived, has undoubtedly unparalleled contribution to algebraic number theory. Much work awaits exploration in the area. In addition, derivation of algebraic results and standardization of Fermat right triangles in deduction of Fermat point and Napoleon points and its association with the centroid of the Napoleon triangle [which in fact is same as the centroid of the basic Fermat right triangle, we started with] is much inspiring and it also leaves wide area for the researchers in the area. We also have extended our work and progressively pursue in the same area and many interesting results are awaited.

REFERENCES

1. Shailesh A. Bhanotar, Ranna A, Trivedi, Dr. Pradeep. J. Jha :International Journal of Applied Research 2015; 1(10): 820-826, IJAR 2015; 1(10): 820-826
2. Uteshev, A. Y., & Yashina, M. V. (2013, September). Stationary points for the family of fermat–torricelli–coulomb-like potential functions. In *International Workshop on Computer Algebra in Scientific Computing* (pp. 412-426). Springer International Publishing.
3. Fermat point-Wikipedia, free encyclopedia-https://en.wikipedia.org/wiki/Fermat_point
4. Fermat points-free wolfram math word- mathworld.wolfram.com/**FermatPoints**.htm
5. Napoleon theorem and Fermat points-<https://www.math.washington.edu/~king/.../12-05-Napoleon-Fermat.html>
6. MOWAFFAQ HAJJA-An Advanced Calculus Approach to Finding the Fermat Point.

Annexure-1 (Addition on the Triplets of the Same Set in Section 1.2-A-(1))

Let $T_{11}, T_{12} \in SD(T_1)$ then

$T_{11} = (m_1/n_1, n_2/m_2, h/(n_1 \times m_2))$ and $T_{12} = (m_1/n_2, n_1/m_2, h/(n_2 \times m_2))$

with basic relation $\left(\frac{m_1}{n_1}\right)^2 + \left(\frac{n_2}{m_2}\right)^2 = \left(\frac{h}{n_1 \times m_2}\right)^2$ and $\left(\frac{m_1}{n_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 = \left(\frac{h}{n_2 \times m_2}\right)^2$.

$T_{11} + T_{12} = (m_1/n_1 + m_1/n_2, n_2/m_2 + n_1/m_2, h/(n_1 \times m_2) + h/(n_2 \times m_2))$

Claim: $\left(\frac{m_1}{n_1} + \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} + \frac{n_1}{m_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} + \frac{1}{(n_2 \times m_2)}\right)^2$

L.H.S. = $\left(\frac{m_1}{n_1} + \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} + \frac{n_1}{m_2}\right)^2$

= $\left(\frac{m_1}{n_1}\right)^2 + \left(\frac{m_1}{n_2}\right)^2 + 2\left(\frac{m_1}{n_1}\right)\left(\frac{m_1}{n_2}\right) + \left(\frac{n_2}{m_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 + 2\left(\frac{n_2}{m_2}\right)\left(\frac{n_1}{m_2}\right)$

= $\left(\frac{m_1}{n_1}\right)^2 + \left(\frac{n_2}{m_2}\right)^2 + \left(\frac{m_1}{n_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 + 2\left(\left(\frac{m_1}{n_1}\right)\left(\frac{m_1}{n_2}\right) + \left(\frac{n_2}{m_2}\right)\left(\frac{n_1}{m_2}\right)\right)$

= $\left(\frac{h}{n_1 \times m_2}\right)^2 + \left(\frac{h}{n_2 \times m_2}\right)^2 + 2\left(\frac{m_1^2 m_2^2 + n_1^2 n_2^2}{n_1 n_2 m_1 m_2}\right)$

= $\left(\frac{h}{n_1 \times m_2}\right)^2 + \left(\frac{h}{n_2 \times m_2}\right)^2 + 2\left(\frac{h^2}{n_1 n_2 m_1 m_2}\right)$

= $h^2 \left(\frac{1}{(n_1 \times m_2)} + \frac{1}{(n_2 \times m_2)}\right)^2$

= R.H.S.

Annexure-2 (Subtraction on the Triplets of the Same Set in Section 1.2-A-(2))

Let $T_{11}, T_{12} \in SD(T_1)$ then $T_{11} = (m_1/n_1, n_2/m_2, h/(n_1 \times m_2))$ and $T_{12} = (m_1/n_2, n_1/m_2, h/(n_2 \times m_2))$

with basic relation $\left(\frac{m_1}{n_1}\right)^2 + \left(\frac{n_2}{m_2}\right)^2 = \left(\frac{h}{n_1 \times m_2}\right)^2$ and $\left(\frac{m_1}{n_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 = \left(\frac{h}{n_2 \times m_2}\right)^2$.

$$T_{11} - T_{12} = (|m_1/n_1 - m_1/n_2|, |n_2/m_2 - n_1/m_2|, |h/(n_1 \times m_2) - h/(n_2 \times m_2)|)$$

$$\text{Claim: } \left(\frac{m_1}{n_1} - \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} - \frac{n_1}{m_2}\right)^2 = h^2 \left(\frac{1}{(n_1 \times m_2)} - \frac{1}{(n_2 \times m_2)}\right)^2$$

$$\begin{aligned} \text{L.H.S.} &= \left(\frac{m_1}{n_1} - \frac{m_1}{n_2}\right)^2 + \left(\frac{n_2}{m_2} - \frac{n_1}{m_2}\right)^2 \\ &= \left(\frac{m_1}{n_1}\right)^2 + \left(\frac{m_1}{n_2}\right)^2 - 2\left(\frac{m_1}{n_1}\right)\left(\frac{m_1}{n_2}\right) + \left(\frac{n_2}{m_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 - 2\left(\frac{n_2}{m_2}\right)\left(\frac{n_1}{m_2}\right) \\ &= \left(\frac{m_1}{n_1}\right)^2 + \left(\frac{n_2}{m_2}\right)^2 + \left(\frac{m_1}{n_2}\right)^2 + \left(\frac{n_1}{m_2}\right)^2 - 2\left(\left(\frac{m_1}{n_1}\right)\left(\frac{m_1}{n_2}\right) + \left(\frac{n_2}{m_2}\right)\left(\frac{n_1}{m_2}\right)\right) \\ &= \left(\frac{h}{n_1 \times m_2}\right)^2 + \left(\frac{h}{n_2 \times m_2}\right)^2 - 2\left(\frac{m_1^2 m_2^2 + n_1^2 n_2^2}{n_1 n_2 m_1 m_2}\right) \\ &= \left(\frac{h}{n_1 \times m_2}\right)^2 + \left(\frac{h}{n_2 \times m_2}\right)^2 - 2\left(\frac{h^2}{n_1 n_2 m_1 m_2}\right) \\ &= h^2 \left(\frac{1}{(n_1 \times m_2)} - \frac{1}{(n_2 \times m_2)}\right)^2 \\ &= \text{R.H.S.} \end{aligned}$$

Annexure - 3 & 4

(Addition and Subtraction on the triplets of different sets in section 1.2-b)

[This can be derived on the same lines as shown in annexure: 1 & 2.]

Conclusions

Different duals of the same Pythagorean triplet or different duals arising from the different sets of the same triplets observe algebraic additive property and satisfy Pythagorean property.

Annexure - 5 (Fermat Point of Right Angle Triangle of Fermat Family)

Let us consider the Fermat right triangle ABC

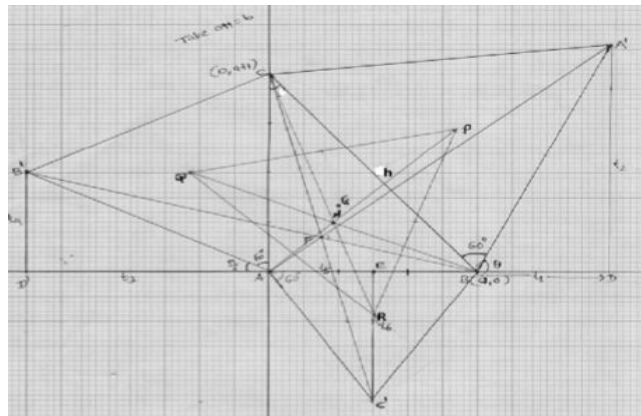


Figure 4: F Fermat Point, N Napoleon Point, and G-Centroid of Napoleon Triangle of Right Triangle of Fermat Family

Let $BC = d, AB = a$ and $AC = b = a + 1$.

$$\therefore \cos B = \frac{a}{h} \& nB = \frac{b}{h} = \frac{a+1}{h}, \text{ where } a, b = a + 1, h = \sqrt{2a^2 + 2a + 1} \in N$$

[It should be noted that h is an integer as our underlying triangle is a right triangle of Fermat family.]

Let $BD = t_1$ & $A'D = t_2$ and $\theta = \angle A'BD$.

$$\therefore \cos \theta = \cos \left(\frac{2\pi}{3} - B \right) = (-1/2)(a/h) + (\sqrt{3}/2)((a+1)/h)$$

$$BD/AB = t_1 / h = (\sqrt{3}(a+1) - a) / 2h$$

$$\therefore t_1 = (\sqrt{3}(a+1) - a) / 2 \text{ or } t_1 = (a(\sqrt{3} - 1) + \sqrt{3}) / 2$$

$$\therefore AD = a + t_1 = a + (a(\sqrt{3} - 1) + \sqrt{3}) / 2 = \frac{(a(\sqrt{3}+1) + \sqrt{3})}{2} \quad (11)$$

$$\& \sin \theta = \sin \left(\frac{2\pi}{3} - \alpha \right) = (\sqrt{3}/2)(a/h) + (1/2)((a+1)/h)$$

$$\therefore A'D/AB = t_2 / h = (\sqrt{3}a + (a+1)) / 2h$$

$$\therefore t_2 = (\sqrt{3}a + (a+1)) / 2 \text{ or } t_2 = (a(\sqrt{3} + 1) + 1) / 2 \quad (12)$$

$$\therefore \text{points of } A' \text{ is } \left(\frac{(a(\sqrt{3}+1) + \sqrt{3})}{2}, \frac{(a(\sqrt{3}+1) + 1)}{2} \right). \quad (13)$$

Let $AD' = t_3$ & $B'D' = t_4$ and $\frac{\pi}{6} = \angle B'AD'$.

$$\cos \left(\frac{\pi}{6} \right) = \frac{t_3}{a+1} \Rightarrow \frac{\sqrt{3}}{2} = \frac{t_3}{a+1} \Rightarrow t_3 = \frac{\sqrt{3}}{2}(a+1)$$

$$\& \sin \left(\frac{\pi}{6} \right) = \frac{t_4}{a+1} \Rightarrow \frac{1}{2} = \frac{t_4}{a+1} \Rightarrow t_4 = \frac{1}{2}(a+1).$$

$$\therefore \text{Points of } B' \text{ is } \left(-\frac{\sqrt{3}}{2}(a+1), \frac{1}{2}(a+1) \right). \quad (14)$$

Note that $\frac{\pi}{3}$ is an angle of $\triangle BAC'$ and let $EC' = t_6$ and $AE = t_5$

$$\therefore \cos \left(\frac{\pi}{3} \right) = \frac{t_5}{a} \Rightarrow t_5 = \frac{a}{2} \& \sin \left(\frac{\pi}{3} \right) = \frac{t_6}{a} \Rightarrow t_6 = \frac{\sqrt{3}a}{2}.$$

$$\therefore \text{Point of } C' \text{ is } \left(\frac{a}{2}, -\frac{\sqrt{3}a}{2} \right). \quad (15)$$

Now we find equations of the lines $AA', BB' \& CC'$.

$$\text{We have, } A = A(0,0), B = B(a,0), C = C(0, a+1), A' = A' \left(\frac{(a(\sqrt{3}+1) + \sqrt{3})}{2}, \frac{(a(\sqrt{3}+1) + 1)}{2} \right),$$

$$B' = B' \left(-\frac{\sqrt{3}}{2}(a+1), \frac{1}{2}(a+1) \right), \text{ and } C' = C' \left(\frac{a}{2}, -\frac{\sqrt{3}a}{2} \right) \quad (16)$$

From these points, we have equation of the lines

$$AA': (a\sqrt{3} + b)x - (b\sqrt{3} + a)y = 0, \text{ where } b = a + 1 \quad (17)$$

$$\mathbf{BB}': bx + (\sqrt{3}b + 2a)y - ab = 0, \text{ where } b = a + 1 \quad (18)$$

$$\mathbf{CC}': (\sqrt{3}a + 2b)x + ay - ab = 0, \text{ where } b = a + 1 \quad (19)$$

To find Fermat point, it is sufficient to find

(I) Intersection point of any two lines. Let us consider the lines \mathbf{BB}' and \mathbf{CC}' . Solving them by Cramer's rule, we have

$$\begin{aligned} \frac{x}{\begin{vmatrix} \sqrt{3}b+2a & -ab \\ a & -ab \end{vmatrix}} &= \frac{y}{\begin{vmatrix} -ab & b \\ -ab & \sqrt{3}a+2b \end{vmatrix}} = \frac{1}{\begin{vmatrix} b & \sqrt{3}b+2a \\ \sqrt{3}a+2b & a \end{vmatrix}} \\ \therefore \frac{x}{-ab(\sqrt{3}b+a)} &= \frac{y}{-ab(\sqrt{3}a+b)} = \frac{1}{-(6ab+2\sqrt{3}h^2)}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2 \\ \therefore x &= \frac{ab(\sqrt{3}b+a)}{(6ab+2\sqrt{3}h^2)} \text{ and } y = \frac{ab(\sqrt{3}a+b)}{(6ab+2\sqrt{3}h^2)} \end{aligned} \quad (20)$$

$$\therefore \text{Fermat point of Fermat family is, } \left(\frac{ab(\sqrt{3}b+a)}{(6ab+2\sqrt{3}h^2)}, \frac{ab(\sqrt{3}a+b)}{(6ab+2\sqrt{3}h^2)} \right), \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2$$

(II) To verify equality of the distances \mathbf{AA}' , \mathbf{BB}' , and \mathbf{CC}' . As we already have the coordinates of the points, we use distance formula.

$$\begin{aligned} d(A, A') &= \sqrt{\left(\frac{a(\sqrt{3}+1)+\sqrt{3}}{2} \right)^2 + \left(\frac{a(\sqrt{3}+1)+1}{2} \right)^2} \\ &= \sqrt{\left(\frac{(a+1)\sqrt{3}+a}{2} \right)^2 + \left(\frac{((a+1)+\sqrt{3}a)}{2} \right)^2}, \text{ take } a+1=b \\ &= \sqrt{\left(\frac{(\sqrt{3}b+a)}{2} \right)^2 + \left(\frac{(b+\sqrt{3}a)}{2} \right)^2} = \sqrt{\frac{4(a^2+b^2)+4\sqrt{3}ab}{4}} \\ &= \sqrt{h^2 + \sqrt{3}ab}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2. \\ d(B, B') &= \sqrt{\left(a - \frac{\sqrt{3}}{2}(a+1) \right)^2 + \left(0 - \frac{1}{2}(a+1) \right)^2} = \sqrt{\left(a - \frac{\sqrt{3}}{2}b \right)^2 + \left(0 - \frac{1}{2}b \right)^2} \\ &= \sqrt{\frac{(2a+\sqrt{3}b)^2+b^2}{4}} = \sqrt{h^2 + \sqrt{3}ab}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2 \\ \& \ d(C, C') &= \sqrt{\left(0 - \frac{a}{2} \right)^2 + \left(b + \frac{\sqrt{3}a}{2} \right)^2} = \sqrt{\frac{a^2+(2b+\sqrt{3}a)^2}{4}} \\ &= \sqrt{h^2 + \sqrt{3}ab}, \text{ where } b = a + 1 \text{ \& } a^2 + b^2 = h^2 \end{aligned}$$

So, finally all these results establish that $\mathbf{AA}' = \mathbf{BB}' = \mathbf{CC}'$. Which is an important property.

Annexure - 6 (Napoleon Point of Right Angle Triangle of a Fermat Family)

From section (1.4), we have $(0,0)$, $B(a,0)$, $C(0,b = a + 1)$, $A' \left(\frac{a(\sqrt{3}+1)+\sqrt{3}}{2}, \frac{a(\sqrt{3}+1)+1}{2} \right)$, $C' \left(\frac{a}{2}, -\frac{\sqrt{3}a}{2} \right)$,

$B' \left(-\frac{\sqrt{3}}{2}(a+1), \frac{1}{2}(a+1) \right)$. Now we derive centroid P, Q & R of equilateral triangles CBA' , $CB'A$ & $AC'B$ respectively.

$$\begin{aligned} \therefore P \left(\frac{\frac{(a(\sqrt{3}+1)+\sqrt{3})}{2}+a+0}{3}, \frac{\frac{(a(\sqrt{3}+1)+1)}{2}+(a+1)+0}{3} \right) &= P \left(\frac{(a+1)\sqrt{3}+3a}{6}, \frac{\sqrt{3}a+3(a+1)}{6} \right) \\ &= P \left(\frac{b+\sqrt{3}a}{2\sqrt{3}}, \frac{\sqrt{3}b+a}{2\sqrt{3}} \right), \text{ where } b = a + 1. \end{aligned} \quad (21)$$

$$Q \left(\frac{0-\frac{\sqrt{3}}{2}(a+1)+0}{3}, \frac{0+\frac{1}{2}(a+1)+(a+1)}{3} \right) = Q \left(\frac{-\sqrt{3}b}{6}, \frac{3b}{6} \right) = Q \left(\frac{-b}{2\sqrt{3}}, \frac{b}{2} \right), \text{ where } b = a + 1. \quad (22)$$

$$\& R \left(\frac{0+\frac{a}{2}+a}{3}, \frac{0-\frac{\sqrt{3}}{2}a+0}{3} \right) = R \left(\frac{3a}{6}, \frac{-\sqrt{3}a}{6} \right) = R \left(\frac{a}{2}, \frac{-a}{2\sqrt{3}} \right). \quad (23)$$

Now we find equation of lines AP , BQ and CR ,

$$AP: (a + 3\sqrt{b})x - (b + 3\sqrt{a})y = 0 \quad (24)$$

$$BQ: 3\sqrt{b}x + (b + 2\sqrt{3}a)y - ab\sqrt{3} = 0 \quad (25)$$

$$CR: (a + 2\sqrt{3}b)x + a\sqrt{3}y - ab\sqrt{3} = 0 \quad (26)$$

Solving by creamer's rule,

$$\begin{aligned} \frac{x}{\begin{vmatrix} b+2\sqrt{3}a & -ab\sqrt{3} \\ a\sqrt{3} & -ab\sqrt{3} \end{vmatrix}} &= \frac{y}{\begin{vmatrix} -ab\sqrt{3} & \sqrt{3}b \\ -ab\sqrt{3} & a+2\sqrt{3}b \end{vmatrix}} = \frac{1}{\begin{vmatrix} \sqrt{3}b & b+2\sqrt{3}a \\ a+2\sqrt{3}b & a\sqrt{3} \end{vmatrix}} \\ \therefore \frac{x}{-ab(\sqrt{3}b+3a)} &= \frac{y}{-ab(\sqrt{3}a+3b)} = \frac{1}{-(10ab+2\sqrt{3}h^2)}, \text{ where } b = a + 1 \& a^2 + b^2 = h^2 \\ \therefore x &= \frac{\sqrt{3}ab(b+\sqrt{3}a)}{(10ab+2\sqrt{3}h^2)} \text{ and } y = \frac{\sqrt{3}ab(a+\sqrt{3}b)}{(10ab+2\sqrt{3}h^2)} \end{aligned}$$

\therefore Napoleon point F of right angle triangle of Fermat family is, $\left(\frac{\sqrt{3}ab(b+\sqrt{3}a)}{(10ab+2\sqrt{3}h^2)}, \frac{\sqrt{3}ab(a+\sqrt{3}b)}{(10ab+2\sqrt{3}h^2)} \right)$, Where $b = a + 1$ and $a^2 + b^2 = h^2$.

And centroid of Napoleon triangle PQR of right angle triangle of Fermat family is

$G = (a/3, b/3)$ where $b = a + 1$ (This can be derived using Vertices P, Q , and R).

